Integrability and Gauge/String Correspondence

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Gauge/String Duality Conjecture

$\mathcal{N} = 4$ super Yang-Mills theory with the gauge group SU(N) is equivalent (dual) to type IIB superstring on $AdS_5 \times S^5$ background

Equivalence means a precise map between the states of the quantized string and the local gauge invariant operators of $\mathcal{N} = 4$ SYM.

The gauge and string theories have the same (super)symmetry group (kinematics!).

Why is it so difficult to verify the conjecture?

- The gauge theory is an exact CFT for all values of the gauge coupling.
  The spectrum of primary operators is hard to determine.

- *The spectrum of quantum string* on the curved background is hard to determine.

I will discuss the recent progress based on ideas of exact integrability.

Plan

1. **Integrable Structure of Gauge Theory**
   - Gauge Theory
   - Spin Chain Interpretation of the Dilatation Operator
   - Bethe Ansatz
   - Higher Loop Integrability

2. **Integrable Structure of String Theory**
   - String Sigma Model
   - Spinning Strings and Neumann Integrability
   - Classical String Bethe Equation
   - (Near) plane-wave quantization

3. **Comparison of Integrable Structures and Beyond**
   - Hidden Commuting Charges for Strings and Spins
   - Quantum String Bethe Equation

4. **Conclusions**

Integrable Structure of Gauge Theory

Content: $\phi^i, A_\mu, \psi^r_\alpha$

$i = 1, \ldots, 6$ and $r = 1, \ldots, 4$.

Action:

$$\begin{aligned} \mathrm{S}_{\mathrm{YM}} &= \frac{1}{g^2_{\mathrm{YM}}} \int d^4x \mathrm{Tr} \Big\{ \frac{1}{4} F^2_{\mu\nu} + \frac{1}{2} (D_\mu \phi^i)^2 + \bar{\psi} \Gamma^\mu D_\mu \psi \\ &- \frac{1}{4} \sum_{ij} [\phi^i, \phi^j]^2 - \frac{i}{2} \Gamma_i [\bar{\psi}, \phi^i] \psi \Big\} \end{aligned}$$

Invariance: Susy+conformal invariance $\rightsquigarrow \mathfrak{psu}(2,2|4)$ algebra

Observables: All gauge-invariant operators transform in UIRs of superconformal group:

$$[\Delta, s_1, s_2; a_1, a_2, a_2]$$

Typical Operators :

$$\mathcal{O}(x) = \mathrm{Tr} \left(\Phi_1^{L-M} \Phi_2^M \right) + \ldots,$$

Here $\Phi_1 = \phi^1 + i\phi^2$, etc. Operators $\mathcal{O}(x)$ realize irrep of SU(4) with labels $[M, L-2M, M]$

Sectors closed under renormalization (mixing):

- **Sector $\mathfrak{su}(2)$.**

  Operators with $[p, q, p]$ and $\Delta_0 = 2p + q$. *Classically they are* $\frac{1}{4}$*-BPS states.* Operators:

$$\mathrm{Tr}(\Phi_1^{L-M}\Phi_2^M) + \dots,$$

  where Φ_1 and Φ_2 are two complex scalars.

- **Sector $\mathfrak{sl}(2)$.** Operators

$$\mathrm{Tr}(\mathcal{D}^M\Phi^L) + \dots,$$

  where $\mathcal{D} = \mathcal{D}_1 + i\mathcal{D}_2$

- **Sector $\mathfrak{su}(1,1)$.** The Dynkin content: $[0, 0, L]$ and $\Delta_0 = \frac{3}{2}L + M$. Operators

$$\mathrm{Tr}(\mathcal{D}^M\Psi^L) + \dots$$

- **Sector $\mathfrak{su}(2|3)$.** Three complex scalars, two complex fermions. *Classically they are* $\frac{1}{8}$*-BPS states.* Operators:

$$\mathrm{Tr}(\Phi_1^{n_1}\Phi_2^{n_2}\Phi_3^{n_3}\Psi_1^{n_4}\Psi_2^{n_5}) + \dots.$$

Spin chains and anomalous dimensions

Renormalization of the composite operators:

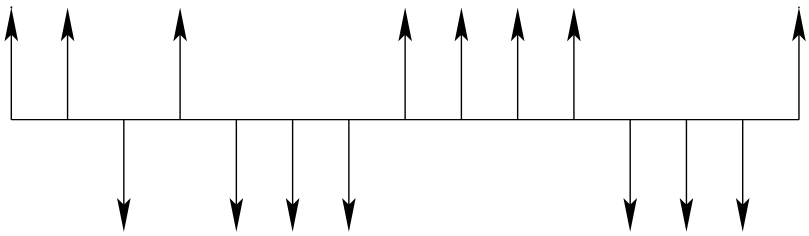
$$\mathcal{O}_I^{\mathrm{ren}} = Z_I{}^J \mathcal{O}_J\,, \qquad D = \frac{dZ}{d\ln\Lambda} Z^{-1}\,, \qquad \Lambda \ \text{is a scale}$$

Eigenvalues of D are anomalous dimensions.

$$\mathrm{Tr}(\Phi_1^{L-M}\Phi_2^M) + \ldots,$$

We refer to the anomalous part of dilatation D as the Hamiltonian $\mathbf{H}$

$$\mathrm{D} = L + g^2\mathbf{H}(g)\,, \qquad g^2 = \frac{\lambda}{8\pi^2} = \frac{g_{\mathrm{YM}}^2 N}{8\pi^2}$$



The (closed) spin chain. The Hamiltonian H acts as $2^L \times 2^L$ matrix, where L is the length of the chain.

M is a number of magnons.

One Loop

At one loop the Hamiltonian of the $\mathfrak{su}(2)$ spin chain is

$$\mathbf{H} = \sum_{i=1}^{L} \Big(I - P_{i,i+1} \Big)$$

This is the famous Heisenberg spin chain.

[Minahan and Zarembo, 2003]

Previously observed integrable structures in QCD
[Lipatov, 1994; Faddeev, Korchemsky, 1995]
[Belitsky, Braun, Derkachov, Korchemsky, Manashov, 1998-2000]
[Kotikov, Lipatov, 2000-2003]

Higher Loops

The dilatation operator on the $\mathfrak{su}(2)$ subsector (up to three loops)

$$\begin{aligned}
\mathbf{H}_{1\ell} &= I - P_{i,i+1} \\
\mathbf{H}_{2\ell} &= -\frac{3}{2} I + 2P_{i,i+1} - \frac{1}{2} P_{i,i+2} \\
\mathbf{H}_{3\ell} &= 5I - 7P_{i,i+1} + 2P_{i,i+2} \\
&- \frac{1}{2}(P_{i,i+3}P_{i+1,i+2} - P_{i,i+2}P_{i+1,i+3})
\end{aligned}$$

This was obtained by careful analysis of the Feynman graphs
Spin chain becomes long-range!

[Beisert, Kristjansen and Staudacher, 2003]

Bethe ansatz and local charges

Bethe equations diagonalize the (one-loop) Hamiltonian

$$\left(\frac{\varphi_k + \frac{i}{2}}{\varphi_k - \frac{i}{2}}\right)^L = \prod_{\substack{j=1 \\ j \neq k}}^{M} \frac{\varphi_k - \varphi_j + i}{\varphi_k - \varphi_j - i}, \qquad \varphi_k \text{ are Bethe roots}$$

[See e.g. Faddeev in Les Houches Proceedings, 1995]

Bethe equations also diagonalize the whole set of local commuting charges!

On a state with M magnons one gets

$$\mathbf{Q}_r = \sum_{k=1}^{M} \mathbf{q}_r(\varphi_k),$$

where $\mathbf{q}_r(\varphi_k)$ are local excitation charges

$$\mathbf{q}_r(\varphi_k) = \frac{i}{r-1}\left(\frac{1}{\left(\varphi_k + \frac{i}{2}\right)^{r-1}} - \frac{1}{\left(\varphi_k - \frac{i}{2}\right)^{r-1}}\right)$$

An eigenvalue of the Hamiltonian is

$$\mathbf{H} = \mathbf{Q}_2 = \sum_{k=1}^{M} \mathbf{q}_2(\varphi_k) = \sum_{k=1}^{M} \frac{1}{\varphi_k^2 + \frac{1}{4}}$$

Particle interpretation of Bethe Ansatz

A magnon is a particle with momentum p_k

$$\exp(ip_k) = \frac{\varphi_k + \frac{i}{2}}{\varphi_k - \frac{i}{2}}$$

The Bethe Ansatz reads

$$\boxed{\exp(iLp_k) = \prod_{\substack{j=1 \\ j\neq k}}^{M} \mathrm{S}(p_k, p_j)\,, \qquad \sum_{k=1}^{M} p_k = 0}$$

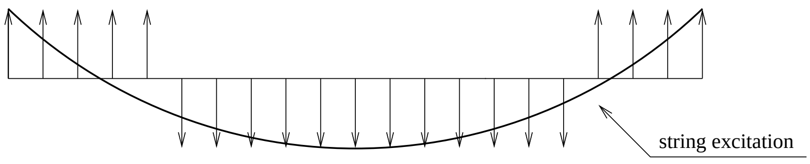
$\mathrm{S}(p_k, p_j)$ is the S-matrix for pairwise scattering of excitations with momenta p_k:

$$\mathrm{S}(p_k, p_j) = \frac{\varphi(p_k) - \varphi(p_j) + i}{\varphi(p_k) - \varphi(p_j) - i}\,, \qquad \underbrace{\varphi(p) = \tfrac{1}{2}\cot(\tfrac{1}{2}p)}_{\text{phase function}}$$

There is an heuristic guess for the all-loop asymptotic Bethe ansatz in the $\mathfrak{su}(2)$ sector but its formulation will be postponed till string theory discussion

"Asymptotic" means that the length of the spin chain (integer!) always exceeds the order of perturbation theory (number of loops).

Gauge/string duality



Basic idea: In the "thermodynamic" limit,

$$M, L \to \infty \quad \text{with} \quad \frac{M}{L} = \alpha \ \text{ fixed}$$

the eigenfunctions of the spin Hamiltonian can be approximated by stationary world-sheet configuration of semiclassical strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$.
Generic eigenvalues should correspond to fluctuations of the quantum string world-sheet.

Comparison

Scaling dimensions of operators in SYM $\Delta(\lambda)$	$\Leftrightarrow$	Energies of classical string configurations $E(\lambda)$

Even more generally

$$\underbrace{\begin{array}{lcl} \text{Infinite tower of local} & & \text{Infinite tower of} \\ \text{commuting charges} & \Leftrightarrow & \text{integrals of motion} \\ \text{in SYM} & & \text{for classical/quantum string} \\ \quad \mathbf{Q}_{\mathrm{gauge}}(\lambda) & & \quad \mathbf{Q}_{\mathrm{string}}(\lambda) \end{array}}_{\text{matching of integrable structures}}$$

String sigma model

- The action is a sum of SO(2,4) and SO(6) sigma models

$$\mathrm{S} = \sqrt{\lambda} \int d\tau d\sigma (L_{\mathrm{S}} + L_{\mathrm{AdS}}) \ ,$$

where

$$\begin{aligned} L_{\mathrm{S}} &= -\frac{1}{2}\partial_a X_i \partial^a X_i + \frac{1}{2}\Lambda(X_i X_i - 1)\,, \\ L_{\mathrm{AdS}} &= -\frac{1}{2}\eta_{ij}\partial_a Y_i \partial^a Y_j + \frac{1}{2}\tilde{\Lambda}(\eta_{ij} Y_i Y_j + 1) \end{aligned}$$

Here

$$X_i, \quad i = 1, \ldots, 6; \qquad Y_i, \quad i = 0, \ldots, 5$$

are the the embedding coordinates; Λ and $\tilde{\Lambda}$ are Lagrange multipliers.

- Virasoro constraints

- Periodicity conditions

$$X(\sigma + 2\pi) = X(\sigma)\,, \qquad Y(\sigma + 2\pi) = Y(\sigma)$$

- Cartan states with only three non-vanishing components

$$J_1 = J_{12}, \quad J_2 = J_{34}, \quad J_3 = J_{56}$$

of the SO(6) angular momentum

$$J_{ij} = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \left(X_i \partial_\tau X_j - X_j \partial_\tau X_i\right) .$$

Neumann System and Rigid Strings

The simplest case: string rotates in S^5 and is trivially embedded in AdS_5 as $Y_5 + iY_0 = e^{i\kappa\tau}$ with $Y_1, \ldots, Y_4 = 0$.

Ansatz for periodic motion with three J_i non-zero:

$$X_1 + iX_2 = x_1(\sigma)\, e^{iw_1\tau}\,,$$
$$X_3 + iX_4 = x_2(\sigma)\, e^{iw_2\tau}\,,$$
$$X_5 + iX_6 = x_3(\sigma)\, e^{iw_3\tau}\,,$$

where

$$X_M^2 = 1 \quad \Longrightarrow \quad \sum_{i=1}^{3} x_i^2 = 1\,.$$

Sigma model evolution equations reduce to the 1-d ("mechanical") system (σ is time now !)

$$x_i'' = -w_i^2 x_i - x_i \sum_{j=1}^{3} \left(x_j'^2 - w_j^2 x_j^2 \right).$$

This is the famous *Neumann dynamical system*

> Sigma model solitons (rigid strings) are described in terms of periodic solutions of the Neumann system

[Frolov, Russo, Tseytlin and G.A., 2003]

The general three spin solutions are described in terms of the hyperelliptic functions. Elliptic two-spin solutions ($x_3(\sigma) = 0$) correspond to folded or circular strings.

Folded string solution

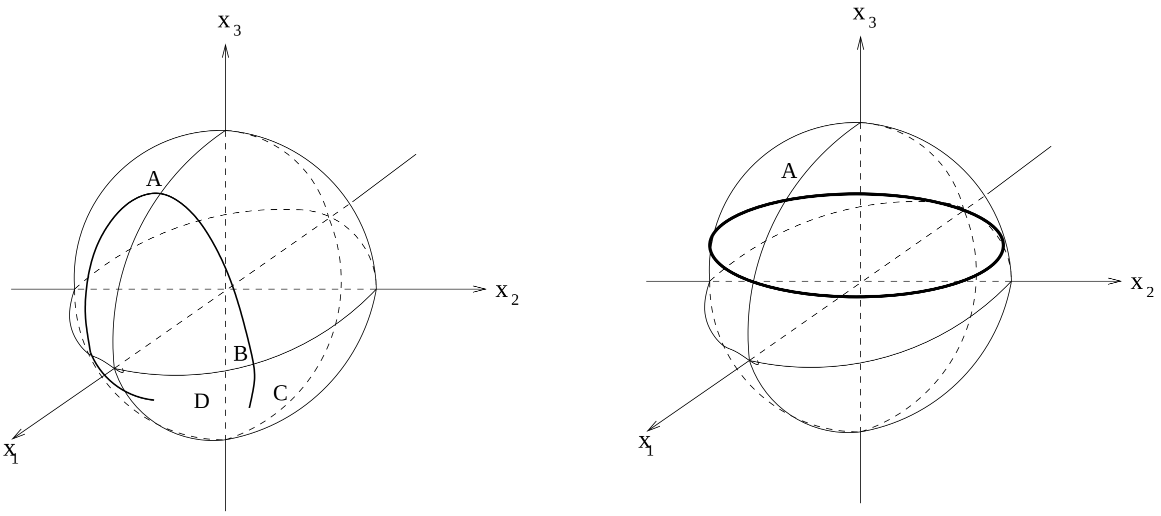
$$x_1(\sigma) = \mathsf{dn}\Big(\sigma\sqrt{w_{21}^2},\ t\Big)\,, \quad x_2(\sigma) = \sqrt{t}\ \mathsf{sn}\Big(\sigma\sqrt{w_{21}^2},\ t\Big)$$

and

$$\frac{\pi}{2}\sqrt{w_{21}^2} = \mathrm{K}(t) \qquad\qquad w_{ij}^2 \equiv w_i^2 - w_j^2$$

Solving for w's in terms of spins J_i, the modulus t can be further found from

$$\left(\frac{J_2}{\mathrm{K}(t)-\mathrm{E}(t)}\right)^2 - \left(\frac{J_1}{\mathrm{E}(t)}\right)^2 = \frac{4}{\pi^2}\lambda\,.$$



Folded and circular rigid strings (generically hyperelliptic).

$$\text{String Energy} \Longrightarrow E = E(J_1, J_2, J_3; \lambda)\,.$$

Classical String Bethe Equation

Strings on $\mathbb{R} \times \mathsf{S}^3$

$$Y_1 = \ldots = Y_4 = X_5 = X_6 = 0; \quad Y_5 + iY_6 = \exp\left(i\frac{E}{\sqrt{\lambda}}\tau\right)$$

$$g = \begin{pmatrix} X_1 + iX_2 & X_3 + iX_4 \\ -X_3 + iX_4 & X_1 - iX_2 \end{pmatrix} \in \mathsf{SU}(2)$$

and make the $\mathfrak{su}(2)$-current

$$A_\tau = g^\dagger \partial_\tau g, \qquad A_\sigma = g^\dagger \partial_\sigma g$$

Construct the x-dependent matrices $\mathscr{L}_\sigma$ and $\mathscr{L}_\tau$:

$$\mathscr{L}_\sigma = \frac{x}{1-x^2}A_\tau + \frac{1}{1-x^2}A_\sigma; \qquad \mathscr{L}_\tau = -\frac{1}{1-x^2}A_\tau - \frac{x}{1-x^2}A_\sigma$$

Zero-curvature (integrability) condition

$$[\mathscr{D}_\tau, \mathscr{D}_\sigma] = 0$$

for $\mathscr{D}_\tau = \partial_\tau - \mathscr{L}_\tau$ and $\mathscr{D}_\sigma = \partial_\sigma - \mathscr{L}_\sigma$ is equivalent to e.o.m.

Monodromy matrix

$$\mathsf{T}(x) = \mathscr{P}\exp\int_0^{2\pi} \mathscr{L}_\sigma(x)\mathrm{d}\sigma$$

Quasi-momentum

$$\mathsf{T}(x) = \begin{pmatrix} e^{ip(x)} & \\ & e^{-ip(x)} \end{pmatrix}$$

Resolvent and Spectral Density

$$\mathbf{G}(x) \equiv p(x) + \frac{E}{\sqrt{\lambda}}\frac{\pi}{x-1} + \frac{E}{\sqrt{\lambda}}\frac{\pi}{x+1} = \int_{\mathbf{C}} \frac{\mathrm{d}x'\rho_{\,\mathsf{s}}(x')}{x-x'}$$

$\mathbf{G}(x)$ is an analytic function on a Riemann surface (x-plane with cuts)

Asymptotics is related to global conserved charges (spins!) $L-M$ and L

$$\mathbf{G}(x) \to \text{no poles}, \qquad x \to \pm 1$$

$$\mathbf{G}(x) \to \frac{2\pi(E-L+2M)}{x\sqrt{\lambda}} + \ldots, \qquad x \to \infty$$

$$\mathbf{G}(x) \to 2\pi m + x\frac{2\pi(E-L)}{\sqrt{\lambda}} + \ldots, \qquad x \to 0$$

On a cut one has an integral equation of the Bethe type

$$2\pi n = \mathbf{G}(x+i0) + \mathbf{G}(x-i0) = 2 \,⨍_{\mathbf{C}} \frac{\rho_{\,\mathsf{s}}(x')}{x-x'}$$

[Kazakov, Marshakov, Minahan and Zarembo, 2003]
General method due to [Smirnov and Reshetikhin, 1983]

Description of finite-gap solutions of the classical string sigma model.

Near plane-wave quantization

- Plane-wave is another maximally supersymmetric background in 10 dim. It is obtained from $\mathrm{AdS}_5 \times \mathrm{S}^5$ by taking the Penrose limit (generalized Wigner contraction)

$$ds^2_{\mathrm{pp}} = 2dx^+dx^- + dz^2 + dy^2 - (z^2 + y^2)(dx^+)^2$$

It is conjectured to be holographically dual to certain subsector in $\mathcal{N} = 4$ SYM

[Berenstein, Maldacena and Nastase, 2002]

- String theory on plane waves is exactly solvable

[Metsaev, Metsaev Tseytlin, 2002]

- The $\mathrm{AdS}_5 \times \mathrm{S}^5$ space (string Hamiltonian) can be recovered by summing up the curvature corrections to the plane wave (string Hamiltonian) background:

$$ds^2 = ds^2_{\mathrm{pp}} +$$
$$+\frac{1}{R^2}\Big[-2y^2dx^-dx^+ + \frac{1}{2}(y^4 - z^4)(dx^+)^2 + (dx^-)^2 + \frac{1}{2}z^2dz^2 - \frac{1}{2}y^2dy^2\Big] + \ldots$$

The first curvature correction to the plane wave energy
[Callan, Lee, McLoughlin, Schwarz, Swanson and Wu, 2003-2004]
(Caltech group)

- Complete bosonic Hamiltonian for (λ, R) finite is known. It is *classically integrable and reproduces the Caltech group results in $1/R$ expansion*

[Frolov and G.A., 2004]

Comparison of Theories

Gauge and string theories exhibit the one and the same integrability structure up to two loops (orders λ and λ^2)! Disagreement starts at three loop order.

Gauge theory	String theory
$\mathbf{Q}_r^{\mathrm{g}}(J_1, J_2, J_3; \lambda)$	$\mathbf{Q}_r^{\mathrm{s}}(J_1, J_2, J_3; \lambda)$
J_i - labels of the h.w.s. of an SO(6) irrep	J_i - spins of a rigid string rotating in S^5
Thermodynamic limit, large J_i	Highly spinning strings, large J_i

On rigid two spin solutions:

$$\mathbf{Q}_r^{\mathrm{s}}(J_1, J_2; \lambda) \stackrel{\lambda, \lambda^2}{=} \mathbf{Q}_r^{\mathrm{g}}(J_1, J_2; \lambda)$$

$r = 1, \ldots, \infty.$

[Staudacher and G.A., 2003]

We found the generating function of local commuting integrals of motion for rigid string solitons by solving the Bäcklund equations for the string sigma model. This generating function was then mapped to the Bethe resolvent of gauge theory.

Gauge and String Theories Have the Same Integrable Structure in the $\mathfrak{su}(2)$ sector up to Two Loops

Bäcklund Transform
of Rigid (Neumann) Strings
[Staudacher and G.A., hep-th/0310182, hep-th/0403077]

Classical String
Bethe Equation
[Kazakov, Marshakov, Minahan and Zarembo, hep-th/0402207]

Ferromagnetic Sigma Model
[Kruczenski, hep-th/0311203; Kruczenski and Tseytlin, hep-th/0406189]

Classical String Bethe Equation

$$\fint_{\mathbf{C}} \frac{d\varphi' \rho_{\,\mathrm{s}}(\varphi')}{\varphi - \varphi'} = \frac{1}{2} \frac{1}{\sqrt{\varphi^2 - 4\omega^2}} + \pi n +$$

$$+ \omega^2 \int_{\mathbf{C}} \frac{d\varphi' \rho_{\,\mathrm{s}}(\varphi')}{\sqrt{\varphi^2 - 4\omega^2}\sqrt{\varphi'^2 - 4\omega^2}} \times$$

$$\times \frac{\varphi - \sqrt{\varphi^2 - 4\omega^2} - \varphi' + \sqrt{\varphi'^2 - 4\omega^2}}{(\varphi + \sqrt{\varphi^2 - 4\omega^2})(\varphi' + \sqrt{\varphi'^2 - 4\omega^2}) - 4\omega^2}$$

CSBE

Spectral density $\rho_{\,\mathrm{s}}(\varphi)$ of a finite-gap solution is normalized as

$$\int_{\mathbf{C}} d\varphi \;\rho_{\,\mathrm{s}}(\varphi) = \frac{M}{L} = \alpha \,.$$

Effective coupling constant ω is $\omega^2 = \frac{g^2}{2L^2}$.

The integers n are winding numbers.

Quantum (discrete) analog of the CSBE? This should describe the *quantum* corrections of the string sigma model.

Quantum String Bethe Equation

The BE we propose to describe the leading quantum effects for strings in the $\mathfrak{su}(2)$ sector

$$\exp(iLp_k) = \prod_{\substack{j=1\\ j\neq k}}^{M} \mathrm{S}(p_k, p_j)\,, \qquad \sum_{k=1}^{M} p_k = 0$$

Here $\mathrm{S}(p_k, p_j)$ is the S-matrix for pairwise scattering of local excitations with momenta p_k

$$\mathrm{S}(p_k, p_j) = \frac{\varphi(p_k) - \varphi(p_j) + i}{\varphi(p_k) - \varphi(p_j) - i} \times$$
$$\times \exp\left(2i \sum_{r=0}^{\infty} \left(\frac{g^2}{2}\right)^{r+2} \mathbf{q}_{r+2}(p_{[k}) \, \mathbf{q}_{r+3}(p_{j]})\right)$$

QSBE

[Frolov, Staudacher and G.A., hep-th/0406256]

$$\mathrm{S}(p_k, p_j) = \frac{\varphi(p_k) - \varphi(p_j) + i}{\varphi(p_k) - \varphi(p_j) - i}$$

Gauge theory ABAE

[Beisert, Dippel and Staudacher, hep-th/0405001]

Phase function $\varphi(p)$ is

$$\varphi(p) = \tfrac{1}{2}\cot(\tfrac{1}{2}p)\sqrt{1 + 8g^2\sin^2(\tfrac{1}{2}p)}\ ,$$

or inverse

$$p := p(\varphi)$$

Remind the relation between g and the 't Hooft coupling λ:

$$g^2 = \frac{\lambda}{8\pi^2}$$

Charges $\mathbf{q}_r(p)$ are

$$\mathbf{q}_r(p) = g^{-r+1}\frac{2\sin(\frac{r-1}{2}p)}{r-1}\left(\frac{\sqrt{1+8g^2\sin^2(\frac{1}{2}p)}\ -\ 1}{2g\sin(\frac{1}{2}p)}\right)^{r-1}$$

In particular,

$$\mathbf{q}_1(p) \quad = \quad p \quad \Leftarrow \quad \text{Momentum}$$

$$\mathbf{q}_2(p) \quad = \quad \frac{1}{g^2}\left(\sqrt{1+8g^2\sin^2(\tfrac{1}{2}p)} - 1\right) \Leftarrow \text{Energy}$$

The total charge of the M excitations given by the sum of individual charges

$$\mathbf{Q}_r = \sum_{k=1}^{M} \mathbf{q}_r(p_k)$$

The energy of string states in the global AdS coordinates is

$$\boxed{\mathbf{E}(g) = L + g^2 \mathbf{Q}_2}$$

Two limiting cases

- Thermodynamic Limit (TDL):
  $M, L \to \infty$ with $\frac{M}{L} = \alpha$ fixed, $\omega^2 = \frac{g^2}{2L^2}$ fixed

- (Near) BMN Limit:
  $M = 2, 3, \ldots$ finite, $L = M + J \to \infty$, $\lambda' = \frac{\lambda}{J^2}$ fixed

Basic Motivation for QSBE:

In the TDL the discrete QSBE reduces to the integral CSBE!

Near-BMN Limit

The BMN limit:

$$M \to \text{finite}, \qquad L = M + J \sim J \to \infty, \qquad \lambda' = \frac{\lambda}{J^2} \to \text{fixed}$$

Scaling in the BMN limit

$$p_k \sim 1/J, \qquad \varphi(p) \sim J$$

No scattering of local excitations in the BMN limit. BEs and their solution

$$e^{ip_k J} = 1 \quad \to \quad p_k = \frac{2\pi n_k}{J}, \qquad \sum_{k=1}^{M} n_k = 0$$

String energy/Anomalous dimension

$$E = \Delta_{\rm g} = J + \sum_{k=1}^{M} \sqrt{1 + \lambda' \, n_k^2}$$

coincides with the BMN energy formula.

The result is the same for both QSBE and the gauge theory ABE.

The near-BMN limit:

One expands (for non-coinciding n_k)

$$p_k = \frac{2\pi n_k}{J} + \frac{p_k^{(2)}}{J^2}$$

and works out the leading large J behavior of the phases

$$\varphi(p_k) = \frac{J}{2\pi n_k}\sqrt{1+\lambda'\, n_k^2} + \mathcal{O}(J^0)$$

and the charges

$$\mathbf{q}_r(p_k) = \frac{2\pi n_k}{J}\left[\frac{4\pi}{\lambda' J}\left(\sqrt{1+\lambda'\, n_k^2}-1\right)\right]^{r-1} + \mathcal{O}(J^{-r-1})$$

The momentum shift

$$\frac{p_k^{(2)}}{2\pi} = -\sum_{\substack{j=1\\ j\neq k}}^{M} \frac{n_k^2\sqrt{1+\lambda'\, n_j^2}+n_j^2\sqrt{1+\lambda'\, n_k^2}}{n_k-n_j}$$

Energy formula for M excitations from QSBE

$$\Delta = J + \sum_{k=1}^{M} \sqrt{1 + \lambda' \, n_k^2}$$
$$- \frac{\lambda'}{J} \sum_{\substack{k,j=1 \\ j \neq k}}^{M} \frac{n_k}{n_k - n_j} \left(n_j^2 + n_k^2 \sqrt{\frac{1 + \lambda' \, n_j^2}{1 + \lambda' \, n_k^2}} \right)$$

Reproduces exactly (!) the cases $M = 2, 3$ of direct quantization near plane-wave background by Callan, Lee, McLoughlin, Schwarz, Swanson and Wu, hep-th/0307032,0404007,0405153 (Caltech group)

Anom. dimension for M excitations from ABAE

$$\Delta_{\mathrm{g}} = J + \sum_{k=1}^{M} \sqrt{1 + \lambda' \, n_k^2} - \frac{\lambda'}{J} \sum_{k=1}^{M} \frac{M \, n_k^2}{\sqrt{1 + \lambda' \, n_k^2}}$$
$$- \frac{\lambda'}{J} \sum_{\substack{k,j=1 \\ j \neq k}}^{M} \frac{2 n_k^2 n_j}{n_k^2 - n_j^2} \left(n_j + n_k \sqrt{\frac{1 + \lambda' \, n_j^2}{1 + \lambda' \, n_k^2}} \right)$$

Strong Coupling Limit $\lambda \to \infty$

We expect to find the famous asymptotics $\sqrt[4]{\lambda}$

Simplest case: $M = 2$

The phase function $\varphi(p) \to \infty$ when $g \to \infty$ limit.

The QSBE reduces to

$$Lp - 8\sqrt{2}\, g\, \cos(\tfrac{1}{2}p)\, \log(\cos(\tfrac{1}{2}p)) = 0$$

The momentum p has an expansion

$$p = \frac{p_0}{\sqrt{g}} + \frac{p_1}{g} + \ldots; \qquad L \ll \sqrt[4]{\lambda}$$

Solution

$$p_0 = 2^{\frac{1}{4}} \sqrt{n\pi}.$$

The leading large λ asymptotics of $E = \Delta_{\mathrm{g}}$ is

$$\Delta_{\mathrm{g}} = 2\left(n^2 \lambda\right)^{\frac{1}{4}}$$

"Gauge theory" operators are dual to string modes
with masses: $m^2 = 4n\sqrt{\lambda}$!

Such a behavior is entirely due to the exponential term in the QSBE!

Summary on QSBE

The Bethe ansatz for quantum strings is proposed.

Properties:

1. In the thermodynamic limit it describes *classical* spinning strings
2. For $M = 2, 3$ it reproduces the $1/J$ correction to string energies found by the Caltech group (near plane-wave quantization) and gives a new prediction for any finite M.
3. At $\lambda \to \infty$ it reproduces the $\sqrt[4]{\lambda}$ behavior of string energies/anomalous dimensions
4. It agrees with gauge theory asymptotic Bethe ansatz up to two loops.
5. It admits interpretation in terms of a long-range spin chain at weak coupling $\lambda \to 0$.[1]

How to *derive* QSBA?

A step forward: Integrable string Hamiltonian for finite λ and J was obtained (Frolov and G.A., hep-th0411089).
In the $1/J$ expansion it reproduces the perturbation theory of the Caltech group!

[1] I have not talked about this.

Conclusions

1. Planar one-loop $\mathcal{N} = 4$ SYM is integrable and has interpretation in terms of PSU(2,2|4) spin chain.

2. Most likely $\mathcal{N} = 4$ SYM is integrable at higher loops. Dilop and the associated Bethe ansatz are known only for the $\mathfrak{su}(2)$ sector.

3. Integrability is not enough to fix the Dilop uniquely.

4. How actually to classify the long-range spin chains?

5. How to account for wrapping interactions? Is integrability only asymptotic?

6. Bethe Ansatz for quantum strings – new interesting approach of getting quantum string spectrum.